

Optimization of a Supersonic Train

Problem Statement

The aim of this project is to design the front end of a supersonic train to minimize a specified cost function. The scope of problem is framed as follows: supersonic flow at state 1 is approaching the front of the train from the left. The cross-section of the front end of the train is composed of a set of points $\{A, B, C_1, \dots, C_N\}$ which lie inside or on the boundary of a cube with side length L as seen in [Figure 1](#). Two points A and B must lie on the right edge of the cube while any number of points C_1 - C_N may construct the rest of the cross-sectional area indicated by the shaded region. The cross section extends uniformly with depth L out of the page to create the volume of the train.

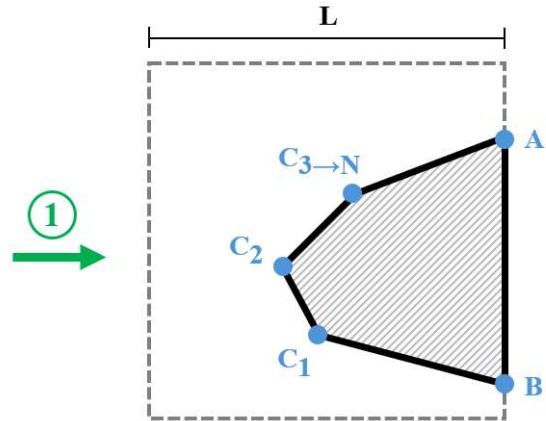


Figure 1: Sketch of Geometry Framework

The cost function to minimize is specified by the mission of the train. The train is transporting L^2 volume from New York to Los Angeles and can only carry the amount of volume enclosed by the geometry of the front end of the train. All the material must be transported and a whole number of trips must be taken. For example, if the volume of the front end is $0.5L^2$, it will take three trips to transport the material: NY→LA (full), LA→NY (empty), NY→LA (full). The design minimizes the cost function $C = 20ND$. C is the cost, N is the number of trips required to transport all the material, and D is the drag of the design in Newtons. The design is to be optimized separately for both inviscid and viscous assumption cases.

The given parameters for the problem are: $M_1 = 3$, $T_1 = 300K$, $P_1 = 101325Pa$, and $L = 1m$.

Approach

Given an arbitrary design, to find the cost one must calculate the number of trips N and the drag D of the body. To calculate the drag on the body, it is necessary to know the pressure forces and velocity of flow surrounding it. The analytical methodology used for determining properties of supersonic flow as it encounters 2D solid bodies are oblique shocks and expansion waves. These equations exist and apply to a specific model of flow which is based on many assumptions. The assumptions place restrictions on the use of these equations and subsequently the design possibilities for this problem. If the design cannot be analyzed using these tools, it will not be considered a valid solution to the problem. Once the flow properties around the body are calculated, drag on the body due to both pressure and viscous forces are readily calculated.

With the cost calculated for an arbitrary design, parameters of a specific shape can be varied to determine local minima. The approach taken in this project is to start with a simple triangle and iteratively make the shape more complex as local minima are found at each iteration.

Initial Assumptions

A few initial assumptions are required of oblique shock and expansion wave theory and can be found described in detail in source 1. The first is that the flow is assumed to be a continuum with respect to the body traveling through it. This means that the number of particles per unit volume is high enough that the flow can be a

continuous media. Fluid properties have no discontinuities as they might at an atomic level. The second assumption made is that the medium is an ideal gas. Intermolecular forces are neglected. This is a reasonable assumption for this problem in which the medium is air at high temperatures and pressures. The third assumption is that the gas is calorically perfect, which means that the specific heats of the gas are constant. The fourth assumption is that the flow is steady, meaning that all derivatives with respect to time are zero. The fifth assumption is that the flow is adiabatic, meaning that there is no heat being added or removed from the flow. Further assumptions will be included as the analysis progresses.

Calculation of Trips

The number of trips required to transport all the material can be found using the following equation which applies the ceiling function to the total material volume divided by the train volume.

$$N = 2[V_{tot}/V_{train}] - 1$$

Oblique Shock Calculation

To calculate the change in flow properties across an oblique shock illustrated in [Figure 2](#), the β - θ - M relation provided by equations 4.19 – 4.21 in source 1 are used to solve for the wave angle beta as a function of the deflection angle theta and Mach number M. The equations are also included below for reference. The term δ is equal to 1 for a weak shock and 0 for a strong shock. For all cases in this paper, it is assumed that the oblique shocks are weak.

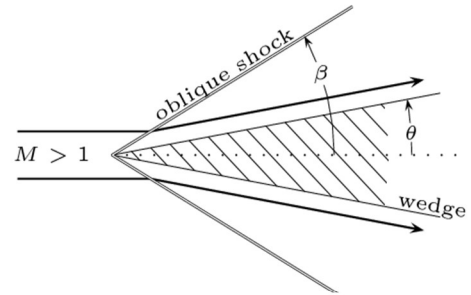


Figure 2: Oblique Shock Diagram

$$\tan \beta = \frac{M^2 - 1 + 2\lambda \cos[(4\pi \delta + \cos^{-1} \chi)/3]}{3 \left(1 + \frac{\gamma - 1}{2} M^2\right) \tan \theta} \quad \lambda = \left[(M^2 - 1)^2 - 3 \left(1 + \frac{\gamma - 1}{2} M^2\right) \left(1 + \frac{\gamma + 1}{2} M^2\right) \tan^2 \theta \right]^{1/2}$$

$$\chi = \frac{(M^2 - 1)^3 - 9 \left(1 + \frac{\gamma - 1}{2} M^2\right) \left(1 + \frac{\gamma - 1}{2} M^2 + \frac{\gamma + 1}{4} M^4\right) \tan^2 \theta}{\lambda^3}$$

From the beta value, the subsequent series of equations are used to determine the flow properties after the shock. These equations are 4.7 – 4.12 in source 1.

$$M_{n_1} = M_1 \sin \beta \quad \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_{n_1}^2}{(\gamma - 1)M_{n_1}^2 + 2} \quad \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_{n_1}^2 - 1)$$

$$M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n_1}^2 - 1} \quad \frac{T_2}{T_1} = \frac{p_2 \rho_1}{p_1 \rho_2} \quad M_2 = \frac{M_{n_2}}{\sin(\beta - \theta)}$$

The calculations in this section are automated using script 1 (see appendix).

Expansion Wave Calculation

To calculate the change in flow properties across an expansion wave illustrated in [Figure 3](#), the Prandtl-Meyer function below is used to find M_2 given the deflection angle θ_2 and M_1 . These equations are 4.44 and 4.45 in source 1.

$$\theta_2 = v(M_2) - v(M_1)$$

$$v(M) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \sqrt{\frac{\gamma - 1}{\gamma + 1} (M^2 - 1)} - \tan^{-1} \sqrt{M^2 - 1}$$

Since the expansion is isentropic, the flow properties after the expansion wave can be found using the standard isentropic relationships below.

$$\frac{T_2}{T_1} = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_2^2} \quad \frac{P_2}{P_1} = \left(\frac{T_2}{T_1} \right)^{\gamma/(\gamma - 1)}$$

The calculations in this section are automated using script 3 (see appendix).

Calculation of Drag of Body

The drag on the front end of the train is due to both pressure and viscous forces. It is assumed that only the X-component of drag impacts the performance of the train (The x axis is aligned with the bottom edge of the cube in [Figure 1](#)). To calculate the drag force efficiently, a suitable control volume analysis is performed. As seen in [Figure 4](#), the control volume is drawn at the boundary of the train body to restrict the terms in the conservation equations to variables found using the techniques currently employed.

The total drag can be found by modifying the momentum conservation equation (Eq. 2.11a, Source 1) seen below.

$$\oint_S (\rho \mathbf{V} \cdot d\mathbf{S}) \mathbf{V} + \oint_{\mathcal{V}} \frac{\partial(\rho \mathbf{V})}{\partial t} d\mathcal{V} = \oint_{\mathcal{V}} \rho \mathbf{f} d\mathcal{V} - \oint_S p d\mathbf{S} + \mathbf{F}_{\text{viscous}}$$

In this case, the first term is equal to zero since there is no mass flow across the surface of the train. Since the model is operating steady-state, the second term is also zero. The third term equals the force required to keep the body stationary, which is equal and opposite to the drag force on the body composed of the fourth and fifth terms.

The fourth term is the pressure drag on the vehicle. At each surface of the vehicle, pressure points inward normal to the surface. Looking at the components in [Figure 5](#), the force in the x direction is $P_i A_i \sin \theta_i$. Notably, it is assumed that the pressure on surface AB in [Figure 1](#) is zero.

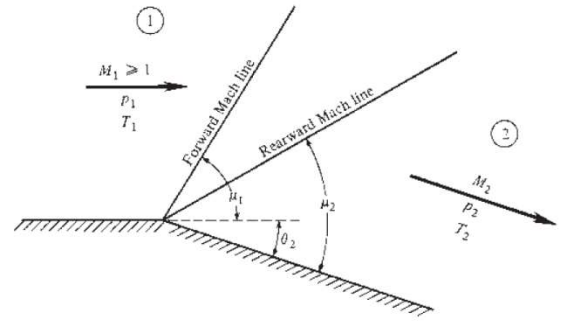


Figure 3: Expansion Wave Diagram (Source 1)

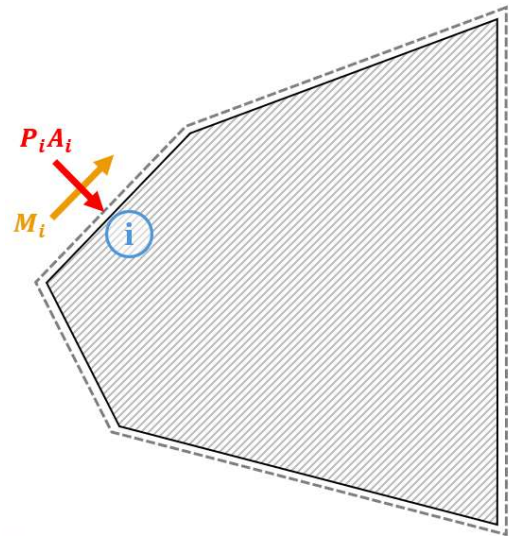


Figure 4: Control Volume Approach

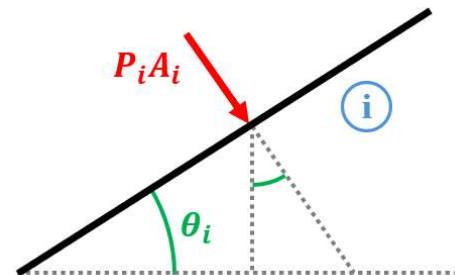


Figure 5: Pressure Component

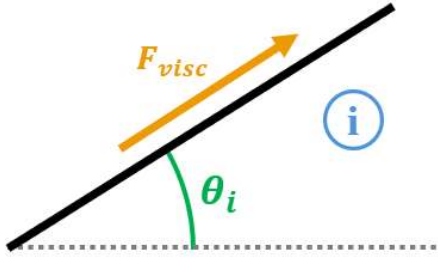


Figure 6: Viscous Component

The fifth term is the viscous forces on the vehicle. In the inviscid case, this term is ignored. At each surface of the vehicle, the viscous forces are equal to the shear stress of the boundary layer at the surface times the area of the surface. The force points tangentially to the surface acting in the direction of the flow. Looking at the components in [Figure 6](#), the force in the x direction is $F_{viscous} \cos \theta_i$. The viscous force can be quantified using the following classic drag equation (Source 3). Using the relationship for an ideal gas $\rho = \frac{P}{RT}$ and

definition of Mach number $M^2 \gamma RT = V^2$ allows the equation to be simplified to terms more readily available in the current methodology.

$$F_{viscous} = 0.5C_d \rho AV^2 = 0.5C_d \gamma PAM^2$$

Simplifying the complete momentum conservation equation leads to the form:

$$D_{total} = D_{pressure} + D_{viscous} = \sum P_i A_i \sin \theta_i + 0.5C_d \gamma \sum P_i M_i^2 A_i \cos \theta_i$$

Initial Parameter Space Reductions

Before calculating any numbers, it is relevant to address a few general trends which reduce the search space for an optimal train. Consider an arbitrary shaped train T with horizontal length x depicted in [Figure 7](#). The length x generally has the following relationships with the surface area and cross-sectional area of the train: $A_{T,surf} \propto x$ and $A_{T,xc} \propto x^2$. This is because as the train gets longer, its thickness at the base increases without inducing further shocks or expansion waves. Drag is linearly proportional to the surface area, while the number of trips is inversely proportional to the cross-sectional area. Plugging in the proportional relationships to the cost function obtains $C \propto x^{-1}$. This basic analysis indicates that maximizing the length of the vehicle will obtain the lowest cost.

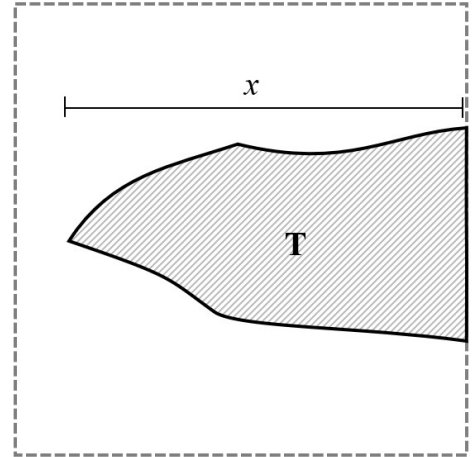


Figure 7: Arbitrary Train

One aspect of the problem that has gone unmentioned is the effect of the ground. The effect is assumed to be negligible, while in reality it would influence the design of the train significantly. This means that the flow approaching the train is symmetric with respect to the mid-plane. Intuitively due to the fact the flow is symmetric, it seems like the problem could be reduced to optimizing one half of the train assuming that it is symmetric. Then an optimal solution for the top half will also be optimal for the bottom.

To test this, the cost of a generic triangle shape seen in [Figure 8](#) with maximum length L is calculated while varying both θ_1 and θ_2 . As can be seen in [Figure 9](#), the minima of the surfaces occur on the line $\theta_1 = \theta_2$. This result gives some weight to the intuitive assumption.

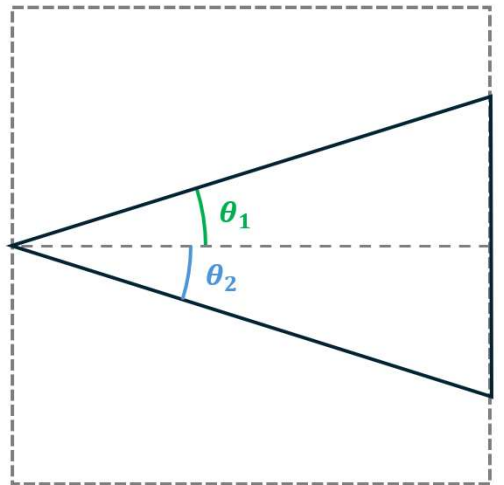


Figure 8: Asymmetric Triangle Parameters

Moving forward the train designs are symmetrical, and the design optimization focuses on one side only. See the appendix for the script used for this calculation.

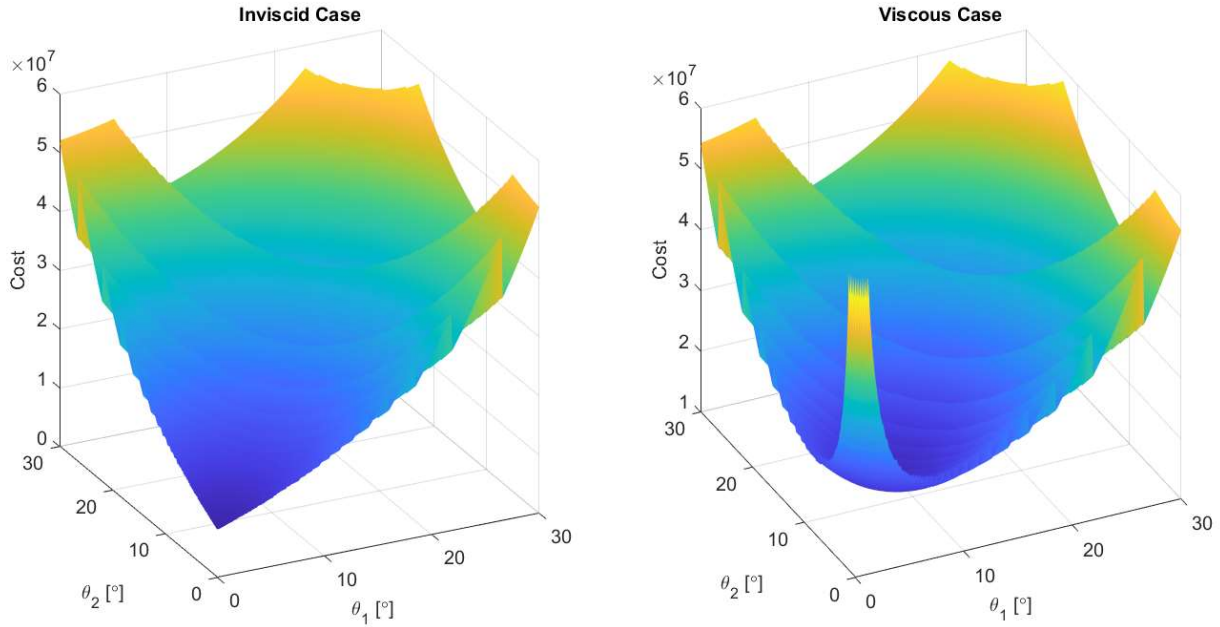


Figure 9: Cost of Asymmetric Triangle

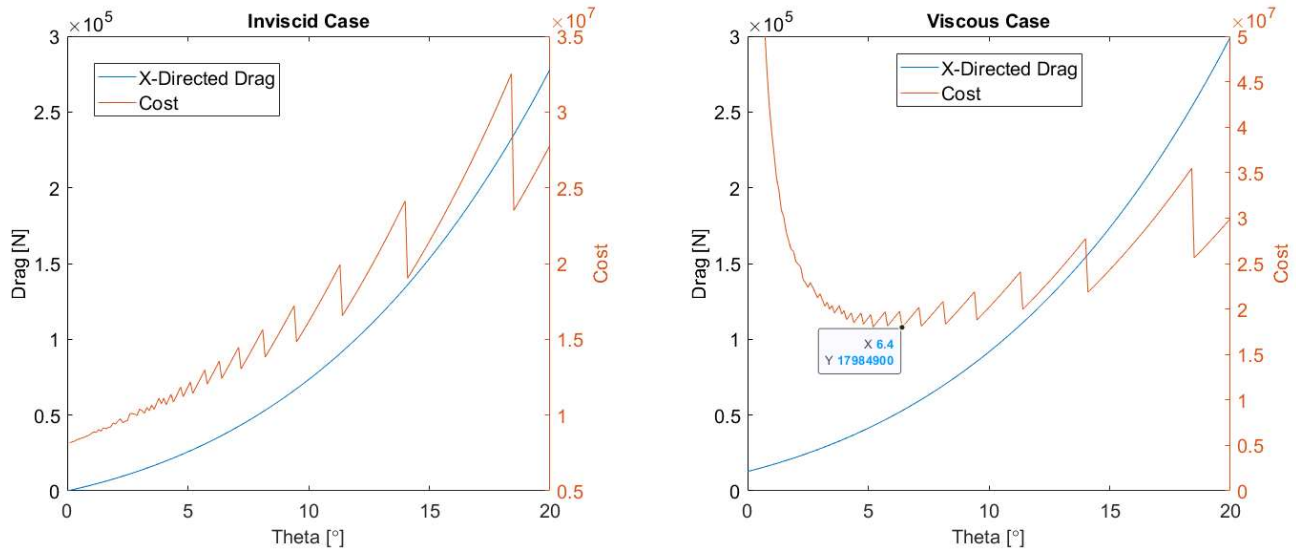


Figure 10: Symmetric Triangle

Case Study: Symmetric Triangle

For a symmetric triangle with parameter $\theta = \theta_1 = \theta_2$ the plot of cost can be seen at the intersection of the plane $\theta_1 = \theta_2$ and the cost surface in [Figure 9](#). Reducing this to one dimension provides [Figure 10](#). While the viscous case has a clear minimum at $\theta = 6.4^\circ$, the inviscid case appears to asymptotically approach some cost value in the limit as the deflection approaches zero. Qualitatively, this means that in the inviscid case as the triangle gets smaller, the drag decreases faster than the number of trips increases. The result seems to indicate paradoxically that the optimal design in the inviscid case is an infinitely thin triangle. The issue with this choice is

that it will violate the continuum assumption essential to the entire analysis. If the train was infinitely thin, it would start “slipping between atoms” (Chacon, 2024). It seems like the inviscid assumption is simply an invalid one to make in this situation.

To provide a definitive answer and satisfy the requirements of this project, the optimized shape of the train front end in the inviscid case is an infinitely thin triangle depicted in [Figure 11](#). While this violates the laws of physics, it is the result of a faulty assumption and perhaps a cost function that is not appropriate for an inviscid situation. The key difference between the inviscid and viscous cases are the viscous forces. In the viscous case, the drag force does not decrease to zero as theta approaches zero due to the friction on the surface area of the vehicle.



Figure 11: Optimal Design - Inviscid

Case Study: Rhombus

Moving forward, only the viscous case will be considered. For the next case study, a rhombus-like shape is optimized through the variation of two parameters θ_1 and θ_2 . The geometric meaning of these parameters is depicted in [Figure 12](#). θ_1 is varied between 0 and 15 degrees at an interval of 0.01. θ_2 is varied between 0 and θ_1 at an interval of 0.01. To do this calculation, the flow properties over section 1 are passed through the expansion wave function called with deflection angle $\theta_1 + \theta_2$.

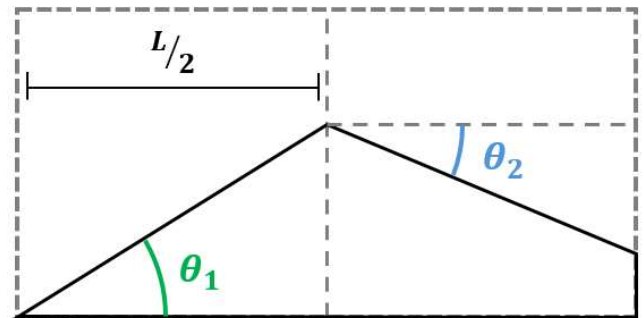


Figure 12: Rhombus Parameters

The resulting cost surface over these two parameters, depicted in [Figure 13](#), has a minimum of $C = 1.452 \times 10^7$ at $\theta_1 = 8.03^\circ$ and $\theta_2 = 1.33^\circ$. Comparing this to the minimum found for a single triangle of $C = 1.80 \times 10^7$ at $\theta = 6.4^\circ$, the cost has decreased while the shape has shifted forward and become thick in the back.

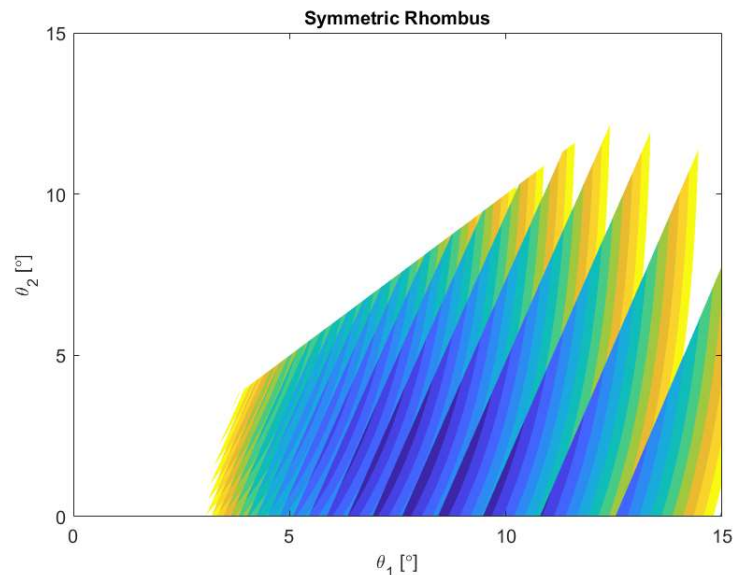


Figure 13: Rhombus Optimization

Final Design

Building off the optimal rhombus shape, two points are added at the x-coordinates $L/4$ and $3L/4$. The y-coordinates vary between 0 and 0.15 for each point while the existing rhombus points stay constant. The cost calculation for this shape involves four different sections each with a different angle, pressure, and Mach number. The contour plot depicting cost as a function of the coordinates y_1 and y_3 is shown in [Figure 14](#). The optimal cost $C = 1.375 \times 10^7$ exists at $y_1 = 0.042m$ and $y_3 = 0.081m$. A rendering of the shape of this train front end is displayed in [Figure 15](#). The drag on this shape is **40.43kN** and it requires **17 trips** to bring all the material from New York to Los Angeles.

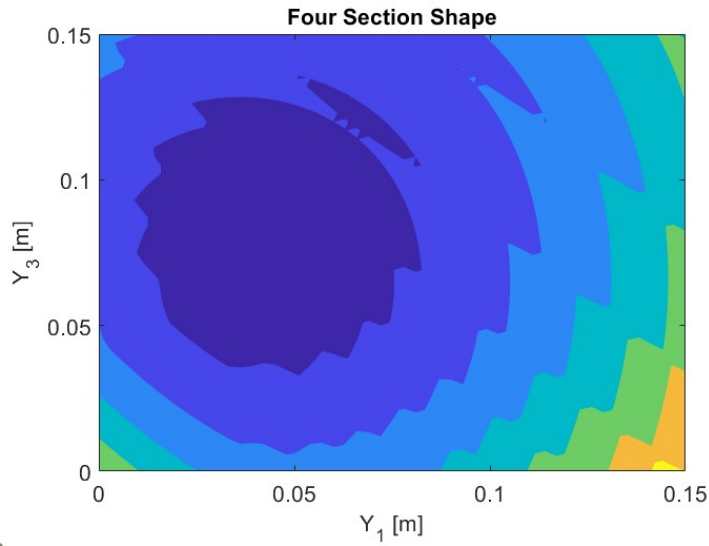


Figure 14: Four Section Shape Optimization

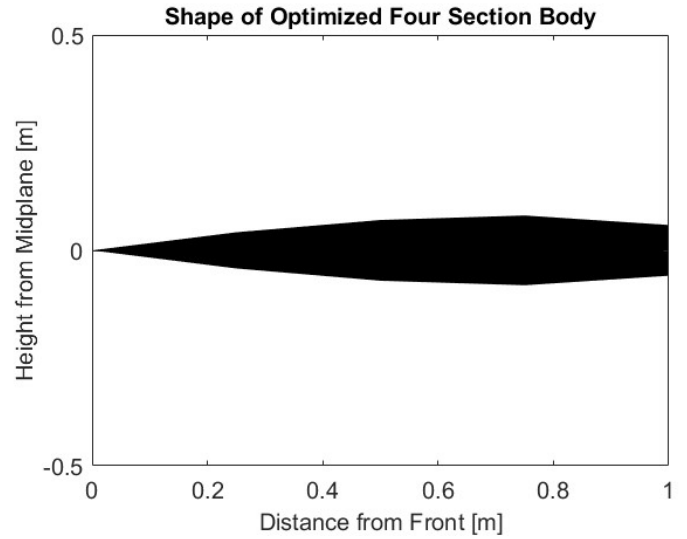


Figure 15: Shape of Four Section Body

Limitations

The optimization approach taken through this work is not ideal because locally optimal parameters for the rhombus shape were propagated forward without any consideration for how optimal they were in the general case of four equally spaced points. In a more thorough study, a form of gradient descent could be applied to a situation in which a greater number of parameters are allowed to vary. The accuracy of such a technique was outweighed in this case by other benefits. Varying two parameters at a time allowed for simplicity in debugging and reduced manual errors.

Works Cited

[1] Anderson, J. (2021). Modern compressible flow: With historical perspective. McGraw-Hill Higher Education.

[2] Apsley, D. (n.d.). Prandtl-Meyer expansion. The University of Manchester.
<https://personalpages.manchester.ac.uk/staff/david.d.apsley/hydraulics/prmeyer.htm>

Used for sanity checking expansion wave calculations.

[3] Benson, T. (n.d.). The drag equation. NASA Glenn Research Center. <https://www.grc.nasa.gov/www/k-12/VirtualAero/BottleRocket/airplane/drageq.html>

[4] Devenport, W. (2023, December 14). Compressible aerodynamics calculator. Virginia Tech Department of Aerospace and Ocean Engineering. <https://devenport.aoe.vt.edu/aoe3114/calc.html>

Used for sanity checking oblique shock calculations.

Appendix

This is a list of the Matlab files I created for this project and what they are used for. They will all be attached in a zip file along with this report.

1. ObliqueShock.m – General oblique shock function.
2. ObliqueShock_MaximumTurningAngle.m – Provides the value of the maximum possible theta for a given Mach number.
3. ExpansionWave.m – General expansion wave function.
4. PrandtlMeyer.m – Evaluates the Prandtl-Meyer function for a given Mach number.
5. RocketSci_SymmetricOpt.m – Used for proving optimality of symmetry in Initial Parameter Space Reductions.
6. RocketSci_SingleAngleOpt.m – Used to calculate optimal angle in Case Study: Symmetric Triangle.
7. RocketSci_RhombusOpt.m – Used to calculate optimal rhombus shape in Case Study: Rhombus.
8. RocketSci_MultiPointOpt_v2.m – Used to calculate optimal four-section shape for final design.